MATH 322 Probability & Statistics
Spring 2011
Midterm 2

1. Suppose that the manufacturer of a new medication wants to test the null hypothesis \( \mu = 0.9 \) against the alternative hypothesis \( \mu < 0.9 \). His test statistic is \( X \), the observed number of successes (recoveries) in 20 trials, and he will accept the null hypothesis in \( x > 14 \); otherwise, he will reject it.

Find \( \alpha \) the Type I error and Type II error \( \beta(0.6) \).

\[
\alpha = P\{\text{Type I error}\} = P\{\text{Reject } H_0 \text{ when } H_0 \text{ is true}\} = P\{X \leq 14, \text{ when } \mu = 0.9\} = P(X \leq 14) \text{ when } X \sim \text{Bin}(20, 0.9) = \text{Pbinom}(14, 20, 0.9) = 0.01125
\]

\[
\beta = P\{\text{Type II error}\} = P\{\text{fail to reject } H_0 \text{ when } H_0 \text{ is false}\} = P\{X > 14, \text{ when } \mu = 0.6\} = 1 - P\{X \leq 14\} \text{ when } X \sim \text{Bin}(20, 0.6) = 1 - \text{Pbinom}(14, 20, 0.6) = 1 - 0.874 = 0.126
\]

2. Suppose that we want to test the null hypothesis that the mean of a normal population \( H_0: \mu = 10 \) vs \( H_a: \mu > 10 \) with \( \sigma^2 = 1 \). Find the value of \( K \) such that \( \bar{x} > K \) provides a critical region of size \( \alpha = 0.05 \) for a random sample of size \( n = 625 \).

\[
\alpha = P\{\text{reject } H_0 \text{ when } H_0 \text{ is true}\} = 0.05 = P\{\bar{x} > K, \text{ when } H_0: \mu = 10\}
\]

\[
0.05 = P\{\bar{x} > K, \text{ when } H_0: \mu = 10\} = P\left\{ Z > \frac{K - 10}{\sqrt{625}} \right\}
\]

\[
(\text{ie}) \quad Z_{0.05} = \frac{K - 10}{\sqrt{625}} = 1.645 = \frac{K - 10}{\sqrt{625}} \implies K = (1.645) \times \frac{1}{\sqrt{625}} + 10 = 10.0658
\]
3. Suppose that we want to test the null hypothesis that the mean of a normal population with \( \sigma^2 = 1 \). Ho: \( \mu = 10 \) vs Ha: \( \mu > 10 \). Determine the minimum sample size needed to test with \( \beta(\mu = 10) = 0.06 \).

\[
 n = \left[ \frac{\sigma [Z_{\alpha} + Z_{\beta}]}{\mu_0 - \mu_1} \right]^2
\]

Let \( \alpha = 0.05 \) \( \Rightarrow \) \( Z_{0.05} = 1.645 \)

\( \beta = 0.06 \) \( \Rightarrow \) \( Z_{0.06} = 1.555 \)

\( \mu_0 = 10 \) and \( \mu_1 = 11 \); \( \sigma = 1 \)

\[
 n = \left[ \frac{1 (1.645 + 1.555)}{10 - 11} \right]^2 = 10.24 
\]

4. A test of a normal population with standard deviation \( \sigma = 5 \) is to be conducted to determine if the true mean is \( \mu = 26 \). It is decided to reject the null hypothesis if a sample of 36 items yields a mean greater than 27.5.

1. State the null and alternative hypotheses for this test.

\[
Ho: \mu = 26 \ \text{vs} \ Ha: \mu > 26
\]

2. If the true population mean is \( \mu = 25.5 \) and the sample is found to have a mean of 28, will the correct decision be made? If not, what type of error will be made?

Ho will be rejected
Type I error will be made.

3. If the true population mean is 26.5 and the sample is found to have a mean of 26.5, will the correct decision be made? If not, what type of error will be made?

Ho is false, Ho will not be rejected
Type II error will be made.

4. What is the level of significance of this test? If the true population mean is 28, what is the power of the test?

\[
\alpha = P\{ \text{Reject } Ho \text{ when } Ho \text{ is true}\}
= P\{ \bar{X} > 27.5 \text{ when } \mu = 26\}
= P\{ Z > \frac{27.5 - 26}{5/\sqrt{36}} \}
= P\{ Z > 1.8 \}
= 0.0359
\]

\[
\text{Power} = P\{ \text{reject } Ho / Ho \text{ is false}\}
= P\{ \bar{X} > 27.5 \text{ when } \mu = 28\}
= P\{ Z > \frac{27.5 - 28}{5/\sqrt{36}} \}
= P\{ Z > -0.6 \}^2
= 0.7257
\]
5. The weights of fish in a certain pond that is stocked regularly are considered to be normally distributed with a standard deviation of 0.5 kg. The following hypothesis will be tested by taking a random sample of 25 fish:

\[ H_0 : \mu = 1.5 \text{ kg} \quad H_a : \mu < 1.5 \text{ kg}. \]

a) If a significance level of 0.02 is to be used, state the decision rule for this test in terms of the sample mean weight of the fish.

\[
P\{ \bar{X} < k \text{ when } \mu = 1.5 \} = 0.02
\]

\[
= P\left\{ Z < \frac{k - 1.5}{0.5/\sqrt{25}} \right\} = 0.02
\]

\[(a) \quad Z_{0.02} = \frac{k - 1.5}{0.5/\sqrt{25}} \Rightarrow k = Z_{0.02} \times \frac{0.5}{\sqrt{25}} + 1.5
\]

\[
= (-2.054) \times \frac{0.5}{\sqrt{25}} + 1.5 = 1.2946
\]

\[
\Rightarrow \bar{X} < 1.2946.
\]

b) If the actual mean weight of fish in this pond is 1.4 kg, what would be

i) the probability of a Type I error?

\[ P\{ \text{Type I error} \} = 0 \text{ since a Type I error is impossible if the null hypothesis is false.} \]

If the true mean is 1.4 kg \( \Rightarrow H_0 \) is false.

ii) the probability of a Type II error?

\[
P = P\{ \text{Type II error} \}
\]

\[
= P\{ \text{fail to reject } H_0 \text{ when } H_0 \text{ is false} \}
\]

\[
= P\{ \bar{X} > 1.2946 \text{ when } \mu = 1.4 \}
\]

\[
= P\left\{ Z > \frac{1.2946 - 1.4}{0.5/\sqrt{25}} \right\} = P\{ Z > -1.06 \}
\]

\[
i \Rightarrow \text{the power of the test} \]

\[
\text{Power} = 1 - P
\]

\[
= 1 - 0.8554 = 0.1446.
\]
6. A drug company claims that more than 80% of the people given a vaccine will develop immunity to a disease. Of 100 randomly picked people who were given the vaccine, 12 did not develop immunity.

1. On the basis of this evidence, is the claim of the drug company valid at a 3% significance level?

\[ H_0: p = 0.8 \]
\[ H_a: p > 0.8 \quad \text{drug company's claim} \]
\[ n = 100 \]
\[ p = 0.88 \]
\[ \alpha = 0.03 \]
\[ Z_\alpha = Z_{0.03} = 1.88 \]

Test statistic:
\[ Z = \frac{p - p_0}{\sqrt{p_0(1-p_0)/n}} \]
\[ = \frac{0.88 - 0.8}{\sqrt{0.8(0.2)/100}} = 2 \]

Reject \( H_0 \) if \( Z \geq Z_\alpha \Rightarrow 2 > 1.88 \)

2. For what levels of significance would you say this sample did not validate the claim?

\[ P\text{-value} = P\{Z \geq \text{test statistic}\} \]
\[ = P\{Z \geq 2.00\} = 1 - P\{Z \leq 2.00\} \]
\[ = 1 - 0.9772 = 0.0228 \]

7. Suppose a TV station wants to determine if more than 50% of the potential TV audience is watching a particular show. The station tells us to design a test such that (i) there is a 2.5% chance of incorrectly concluding that more than 50% are watching and (ii) a 3% chance of concluding no more than 50% are watching when actually 60% of the potential audiences are watching. How large a sample is required?

\[ H_0: p = 0.5 \]
\[ H_a: p > 0.5 \]
\[ \alpha = \text{Type I error} \]
\[ = \text{incorrectly rejecting } H_0 \]
\[ = 0.025 \]
\[ \beta(0.6) = 0.03 \]

\[ Z_\alpha = Z_{0.025} = 1.96 \]
\[ Z_\beta = Z_{0.03} = 1.88 \]
\[ p = 0.5 \quad p' = 0.6 \]

\[ p_0' = \frac{p_0 + Z_\alpha^2 + Z_\beta^2}{2} = \frac{0.5 + 1.96^2 + 1.88^2}{2} \]
\[ = \frac{0.5 + 3.8416 + 3.5344}{2} \]
\[ = \frac{7.976}{2} \]
\[ = 3.988 \]

\[ n = \left[ \frac{Z_\alpha \sqrt{p_0' (1-p_0')} + Z_\beta \sqrt{p_0' (1-p_0')}}{p_0' - p_0} \right]^2 \]
\[ = \left[ \frac{1.96 \sqrt{0.5503} + 1.88 \sqrt{0.3503}}{0.6 - 0.5} \right]^2 \]
\[ = 361.38 \]
\[ \approx 362 \]
8. Eleven sections of bacon each weighing 100 grams and consisting of both back fat and lean muscle tissue were homogenized in a food processor and then chemically treated. Their copper content was then assayed by submitting the samples for copper analysis by flame atomic absorption spectroscopy. The following data gives the copper content of the fat (g/g):

\[ 0.55 \ 0.46 \ 0.72 \ 0.32 \ 0.45 \ 0.47 \ 0.36 \ 0.62 \ 0.71 \ 0.51 \ 0.39 \]

\[ \bar{x} = 0.505 \]
\[ s = 0.1335 \]

1. At the 2% level is the true standard deviation significantly different from 0.12?

\[ H_0 : \sigma = 0.12 \]
\[ H_a : \sigma \neq 0.12 \]

Decision Rule: \[ \chi^2 \geq \chi^2_{0.01, n-1} \]

\[ \chi^2 = \frac{(n-1)s^2}{\sigma^2} \]

\[ \chi^2 = \frac{(10)(0.1335)^2}{(0.12)^2} = 12.38 \]

Fail to reject \( H_0 \)

2. For what levels of significance would you conclude the standard deviation was significantly different from 0.12?

\[ P-value = 2 \times P\{ \chi^2_{10, \alpha/2} \geq 12.38 \} \]

\[ = 2 \times [0.2604] = 0.5208 \]

\[ \Rightarrow H_0 \ text{ would be rejected for significance levels greater than } 0.5208 \]

3. Test the claim that the true mean copper content is greater than 0.5. Use a 1% significance level.

\[ H_0 : \mu = 0.5 \]
\[ H_a : \mu > 0.5 \]

Decision Rule: \[ t \geq t_{0.01, n-1} \]

\[ t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \]

\[ t = \frac{0.505 - 0.5}{0.1335/\sqrt{11}} = 0.13549 \]

Do not reject \( H_0 \).

4. What is the p-value of the test statistic in 3)?

\[ P-value = P\{ t_{10, d.f} > 0.13 \} = 0.55 \]
9. A cigarette manufacturer is experimenting with two new filters for its cigarettes. Filter I is cheaper to make than Filter II and will be used unless there is statistical evidence that Filter II is more effective than Filter I. To compare the two filters, 25 cigarettes were made using each type of filter. The cigarettes were mechanically smoked and the amount (in milligrams) of trapped tar and nicotine recorded with the following results:

<table>
<thead>
<tr>
<th></th>
<th>Filter I</th>
<th>Filter II</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>1.10</td>
<td>1.13</td>
</tr>
<tr>
<td>Variance</td>
<td>.05</td>
<td>.07</td>
</tr>
</tbody>
</table>

Assume the amounts of trapped tar and nicotine for cigarettes made with both filters are normally distributed with common variances.

a) At a 5% significance level, which filter do you think the manufacturer should choose?

Given: \( n_1 = 25, \bar{x}_1 = 1.10, s_1^2 = 0.05, n_2 = 25, \bar{x}_2 = 1.13, s_2^2 = 0.07 \)

Pooled variance: \( s_p^2 = \frac{(24)(0.05) + (24)(0.07)}{48} = 0.06 \)

\( H_0: \mu_1 - \mu_2 = 0 \)
\( H_a: \mu_1 - \mu_2 < 0 \)

Decision Rule: \( t \leq -t_{\alpha, d.f.} \)

\( t_{0.05, 48} = 1.677 \)

Test statistic: \( t = \frac{(\bar{x}_1 - \bar{x}_2) - \Delta_0}{s_p \sqrt{\left( \frac{1}{n_1} + \frac{1}{n_2} \right)}} \)

\( = -0.433 \)

Do not reject \( H_0 \)

\( \Rightarrow \) there is no significance difference.

b) Noting that Filter I is cheaper, do you think the manufacturer would prefer a greater risk for making a Type I error or a Type II error in the test in a)? Explain

The manufacturer would probably prefer a greater risk as a Type II error— that is, they would rather risk using Filter I (which is cheaper) when Filter II was actually more effective.

A Type I error would result in using the more expensive Filter II when it was no more effective than Filter I.
10. Suppose a television network wishes to determine whether major sports events attract more viewers than first-run movies in the prime-time hours. It selects 28 prime-time evenings; of these, 13 have programs devoted to major sports events and the remaining 15 have first-run movies. A television viewer rating firm is employed to obtain random samples and produces the following results:

<table>
<thead>
<tr>
<th></th>
<th>SPORTS</th>
<th>MOVIES</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>6.8 million</td>
<td>5.3 million</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>1.8 million</td>
<td>1.6 million</td>
</tr>
<tr>
<td>Sample Size</td>
<td>13</td>
<td>15</td>
</tr>
</tbody>
</table>

The rating firm suggests the number of viewers may be considered to normally distributed and the standard deviations in the number of viewers of sports and of movies are essentially the same.

a) Calculate the best estimate of this common population standard deviation.

\[
S_p = \sqrt{\frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1 + n_2 - 2}} = \sqrt{\frac{(12)(1.8)^2 + (14)(1.6)^2}{13 + 15 - 2}} = (1.6952)
\]

b) Calculate the test statistic for the hypothesis test which the network is conducting.

\[
t = \frac{(\bar{x} - \bar{y}) - 0}{\sqrt{S_p^2 \left( \frac{1}{n_1} + \frac{1}{n_2} \right)}} = \frac{1.5}{\sqrt{(1.6952)(\frac{1}{12} + \frac{1}{15})}} = 2.335
\]

c) What should the network conclude about which programming attracts more viewers? Why?

Using tables of the t-distribution with 26 df,

\[0.01 < P\text{-value} < 0.02\]

If \( P \text{-value} < 0.01 \) \( \Rightarrow \) no difference in the number of viewers
If \( P \text{-value} > 0.02 \) \( \Rightarrow \) sporting events attract more viewers.

d) What type of error might be made in your conclusion in c)

If we decide sports attract more viewers (\( P \geq 0.02 \))

a type I error have been committed.

If we conclude that the two are same (\( P \leq 0.01 \))

a type II error may have been made.
11. A tire company wants to test, at the 5% significance level, that the tread wear on a new type of tire will be less than the tread wear on an old model in 45,000 kilometers. Ten cars are used for the test. The left front wheel of each car will carry the new design; the right front wheel will carry the old design. The following data represents the number of millimeters of tread wear:

<table>
<thead>
<tr>
<th>Car</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Old Design</td>
<td>1.4</td>
<td>2.1</td>
<td>1.7</td>
<td>2.9</td>
<td>1.0</td>
<td>3.4</td>
<td>2.5</td>
<td>1.7</td>
<td>2.4</td>
<td>2.0</td>
</tr>
<tr>
<td>New Design</td>
<td>1.2</td>
<td>2.0</td>
<td>1.7</td>
<td>2.6</td>
<td>1.1</td>
<td>2.5</td>
<td>2.3</td>
<td>1.5</td>
<td>2.1</td>
<td>1.6</td>
</tr>
</tbody>
</table>

\[ d_i = X_{old} - X_{new} \]

\[ d = 0.25 \quad S_d = \frac{0.27182}{\sqrt{10}} \]

\[ H_0: \mu_d = \mu_{old} - \mu_{new} = 0 \]
\[ H_a: \mu_d = \mu_{old} - \mu_{new} > 0 \]

Decision Rule: \( t > t_{0.05, 9} \)

Test Statistic: \[ t = \frac{d - \mu_d}{S_d / \sqrt{n}} = \frac{0.25}{0.27182 / \sqrt{10}} = 2.908 \]

\[ t_{0.05, 9} = 1.833 \]

[Reject \( H_0 \)]

Reject \( H_0 \) and conclude the tread wear on the new type of tire is significantly less than the old.

12. A market research firm is interested in determining if the proportion of Luther College students who own a car is the same as that for Grinnell College students. They interview 240 Luther students and find 78 who do not own a car. From the 270 Grinnell students interviewed at the college, 82 do not own a car.

a) At a 2% level of significance, is there a difference in the proportion of car-owners at the two institutions?

Let success be "owning a car"

For Luther proportion: \( \frac{162}{240} = 0.675 \)

For Grinnell proportion: \( \frac{188}{270} = 0.6963 \)

Pooled estimate \( \hat{p} = \frac{X + Y}{m + n} = \frac{350}{510} = 0.68627 \)

\[ H_0: P_1 - P_2 = 0 \]
\[ H_a: P_1 - P_2 \neq 0 \]

Decision rule: \[ Z > Z_{0.02} \quad (or) \quad Z < -Z_{0.02} \]

Test statistic: \[ Z = -0.517 \quad \text{and} \quad Z_{0.01} = 2.326 \]
Do we reject $H_0$ and conclude the proportion of students who own a car is the same at both schools?

b) Determine the p-value of the test in a).

$$P\text{-value} = 2 \times P\{ Z > |\text{test statistic}| \}$$

$$= 2 \times [1 - \Phi(1.21)]$$

$$= 2 \times [1 - 0.5117]$$

$$= 2 \times [1 - 0.6985] = 0.603.$$  

13. Tablets produced by pharmaceutical firms are supposed to contain a consistent amount of active ingredient. Firm A claims its tablets have less variability in the amount of active ingredient than those of its competitor, Firm B. Random samples from the two firms showed these amounts of active ingredient:

<table>
<thead>
<tr>
<th>Firm A</th>
<th>9.1</th>
<th>9.0</th>
<th>9.5</th>
<th>8.7</th>
<th>10.1</th>
<th>9.3</th>
<th>9.2</th>
<th>9.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Firm B</td>
<td>8.1</td>
<td>7.5</td>
<td>8.9</td>
<td>9.5</td>
<td>7.1</td>
<td>6.9</td>
<td>8.0</td>
<td>7.8</td>
</tr>
</tbody>
</table>

a) Test Firm A's claim at 5% level of significance.

Firm A: $n_A = 8$, $\bar{x} = 9.3375$, $s_1 = 0.4501$

Firm B: $n_B = 11$, $\bar{y} = 8.1727$, $s_2 = 0.833$

$H_0: \sigma_1^2 = \sigma_2^2 \quad [\frac{\sigma_1^2}{\sigma_2^2} = 1]$

$H_a: \sigma_1^2 < \sigma_2^2 \quad [\frac{\sigma_1^2}{\sigma_2^2} < 1]$

Decision Rule: $F < F_{1-\alpha, 4, 10}$

Test Statistic $F = \frac{s_1^2}{s_2^2} = \frac{0.20268}{0.69418} = 0.29197$

$0.292 > 0.275 \Rightarrow$ fail to reject $H_0$.

b) Find the P-value of the test in a)

There is no evidence to support Firm A's claim.

$P\text{-value} \text{ between 0.05 \& 0.1}$
c) Construct a 98% confidence interval estimate of the true standard deviation of Firm A's tablets.

\[
\frac{150 (1-x) \%}{1 - x} \text{ for } \sigma^2 \text{ is } \left[ \frac{(n-1) s^2 / \chi^2_{1-\frac{x}{2}, n-1}}{\chi^2_{\frac{x}{2}, n-1}} , \frac{(n-1) s^2 / \chi^2_{1-\frac{x}{2}, n-1}}{\chi^2_{\frac{x}{2}, n-1}} \right] \\
\chi^2_{0.01,7} = 18.4753 \\
\chi^2_{0.99,7} = 10.239043
\]

\[\alpha = 2\% \]
\[\alpha' = 1\%\]

\[0.07678 \leq \sigma^2 \leq 1.14502\]
\[0.27710 \leq \sigma^2 \leq 1.07006\]

d) Construct a 98% confidence interval estimate of the true mean difference in active ingredient in the tablets from the two firms.

If you assume equal variance
\[\hat{s}^2 = 0.4916\]
\[\bar{X} - \bar{\bar{X}} = t_{\frac{x}{2}, n-1} \sqrt{\hat{s}^2 \left( \frac{1}{m} + \frac{1}{n} \right)} \]
\[(0.3283, 2.001)\]

\(t_{0.01,7} = 2.567\)

14. Let \(X_1, X_2, \ldots, X_n\) be a random sample of size \(n = 19\) from the normal distribution \(N(\mu, \sigma^2)\).

a) Find the rejection region of size \(x = 0.05\) for testing

\(H_0: \sigma^2 = 30\) vs \(H_1: \sigma^2 > 30\).

For which values of the sample variance \(s^2\) should the null hypothesis be rejected?

Test Statistic: \(\chi^2 = \frac{(n-1)s^2}{\sigma^2} = \frac{18.8^2}{30}\)

Reject \(H_0\) if \(\chi^2 > \chi^2_{0.05,18} = 28.87\)

So \(18.8 > 28.87 \Rightarrow s^2 > 48.1166\)

b) What is the probability if Type II Error for the rejection region when \(\sigma^2 = 80\).

\[P(\text{Type II error}) = P(\text{Accept } H_0 \text{ if } H_0 \text{ is false})\]

\[= P\left( s^2 \leq 48.1166 \text{ when } \sigma^2 = 80 \right)\]

\[= P\left( \frac{(n-1)s^2}{\sigma^2} < \frac{48.1166(n-1))}{80} \right)_{10}\]

\[= P\left( \chi^2 < 10.8262 \right) \approx 0.10 \]
15. A corporation owns a large apple orchard that is leased to an operator. The lease requires the operator to spray the orchard (at the operator’s expense) for a certain leaf disease each spring. However, the requirement will be waived if the operator can provide evidence that the percentage of trees in the orchard which are diseased is less than 4%. The operator took a random sample of 200 trees in the orchard and found 5 diseased trees.

a) At a 2% level of significance, could it be concluded the true percentage of diseased trees was less than 4% so the requirement would be waived?

\[ \text{Ho: } p = 0.04 \]
\[ \text{Ha: } p < 0.04 \]
\[ p = \frac{5}{200} = 0.025 \]

Rejection region: \( Z \leq -2.05 \)
\( \alpha = 0.02 \)
\[ Z_{0.02} = -2.05 \]

Test statistic:
\[ Z = \frac{0.025 - 0.04}{\sqrt{(0.04)(0.96)/200}} = -1.08 \]

\(-1.08 > -2.05 \)

Fail to reject Ho \( \Rightarrow \) there is no evidence that \% of diseased trees are less than 4%.

b) If only 3% of trees in the orchard which are diseased, how likely is it that a level 0.01 test based on \( n = 150 \) trees will detect such a departure from Ho? And what should be the sample size be to ensure that \( \beta(0.03) = 0.01 \), with \( \alpha = 0.01 \)?

\[ \beta(0.03) = 1 - \Phi \left[ \frac{0.03 - 0.04 - 2.33 \sqrt{(0.04)(0.96)/150}}{\sqrt{(0.03)(0.97)/150}} \right] \]
\[ = 1 - \Phi \left[ -2.33\sqrt{(0.04)(0.96)/150} \right] \]
\[ = 1 - 0 = 1 \]

\[ n = \left[ \frac{2.33 \sqrt{(0.04)(0.96)} + 2.33 \sqrt{(0.03)(0.97)}}{0.03 - 0.04} \right] \]
\[ = 7295 \]