NAME---------------------------------------------
MATH 321 Exam #3 100 Points Nov 25, 2013

You must show enough work, or give sufficient explanation, in each problem to clearly indicate how you obtain your answer. No credit will be given for a problem if there is insufficient work/explanation.

1. A company that services air conditioner units in residences and office blocks is interested in how to schedule its technicians in the most efficient manner. The random variable X, taking the values 1, 2, 3 and 4, is the service time in hours. The random variable Y, taking the values 1, 2 and 3, is the number of air conditioner units. The joint probability mass function is given in the following table.

<table>
<thead>
<tr>
<th>Y= # of Units</th>
<th>X=1</th>
<th>X=2</th>
<th>X=3</th>
<th>X=4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.12</td>
<td>0.08</td>
<td>0.07</td>
<td>0.05</td>
</tr>
<tr>
<td>2</td>
<td>0.08</td>
<td>0.15</td>
<td>0.21</td>
<td>0.13</td>
</tr>
<tr>
<td>3</td>
<td>0.01</td>
<td>0.01</td>
<td>0.02</td>
<td>0.07</td>
</tr>
</tbody>
</table>

Find Conditional distribution of X=1 / Y=3. [5]

\[ P(X=1 / Y=3) = \frac{P(X=1, Y=3)}{P(Y=3)} = \frac{0.01}{0.11} = 0.091 \]

2. Roll a die until we get a 6. Let Y be total number of rolls and X the number of 1’s we get. Compute E[X/Y=y]. [5]

The event Y = y means that there were y - 1 rolls that were not 6 and then 6-th roll was a "6."

So, the distribution of X/y = y ~ Binomial (n = y - 1, p = 1/6)

\[ E(X/y = y) = (y-1) \cdot \frac{1}{6} \]

3. Suppose that a rectangle is constructed by first choosing its length, X and then choosing its width Y. Its length X is selected form an exponential distribution with mean \( \mu = 5 \). Once the length has been chosen its width, Y, is selected from a uniform distribution from 0 to half its length. Find the joint pdf of X&Y. [5]

\[ f_X(x) = \frac{1}{5} e^{-\frac{1}{5}x} \quad \text{for} \quad x \geq 0 \]

\[ f_Y(y/x=x) = \frac{1}{x/2} \quad 0 \leq y \leq x/2 \]

\[ f(x,y) = f(y/x=x) \cdot f_x(x) \]

\[ = \frac{1}{x/2} \cdot \frac{1}{5} e^{-\frac{1}{5}x} = \frac{2}{5x} e^{-\frac{1}{5}x} \quad 0 \leq y \leq \frac{x}{2} \]

\[ x \geq 0 \]
4. A bird lands in a grassy region described as follows: Let $X$ and $Y$ be the coordinates of the bird's landing. Assume that $X$ and $Y$ have the joint density

$$f_{XY}(x, y) = \begin{cases} \frac{1}{50} & \text{for } x \geq 0, \; y \geq 0, \; x + y \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

Find $P(X \leq 7 \text{ and } Y \leq 7)$. [10]

$$P(X \leq 7, Y \leq 7) = \int_{0}^{7} \int_{0}^{10-x} \frac{1}{50} \, dy \, dx$$

(O.R.)

$$P(X \leq 7, Y \leq 7) = 1 - P(X > 7 \text{ or } Y > 7)$$

$$= 1 - \left[ P(X > 7) + P(Y > 7) - P(X > 7 \text{ and } Y > 7) \right]$$

$$P(X > 7) = P(Y > 7) = \int_{7}^{10} \int_{0}^{5-x} \frac{1}{50} \, dy \, dx = \frac{4.5}{50} = 0.09$$

$$= 1 - \left( \frac{4.5}{50} - \frac{4.5}{50} \right)$$

$$= 41/50$$

5. Freddy and Jane have entered a game in which they each win between 0 and 2 dollars. If $X$ is the amount Freddy wins, and $Y$ is the amount that Jane wins, they believe that the joint density of their winnings will be

$$f_{XY}(x, y) = \begin{cases} \frac{1}{4} & \text{for } 0 \leq x \leq 2, \; 0 \leq y \leq 2, \\ 0 & \text{otherwise} \end{cases}$$

Find the probability that their combined winnings exceed 2. [10]

$$P(X + Y > 2) = \int_{0}^{2} \int_{0}^{2-x} \frac{1}{4} \, dy \, dx$$

$$= \int_{0}^{2} \frac{1}{4} \left[ y \right]_{y=0}^{y=2-x} \, dx$$

$$= \int_{0}^{2} \frac{1}{4} \left[ (2-x) - 0 \right] \, dx$$

$$= \int_{0}^{2} \frac{1}{4} (2-x) \, dx$$

$$= \left[ \frac{1}{8} \left( x^2 - 2x \right) \right]_{x=0}^{x=2}$$

$$= \frac{1}{8} \left( 4 \left( \frac{3}{2} \right) - 2 \times \frac{4}{4} \right) = 5/6$$

$$= 0.833$$
6. Every day, a student calls his mother and then (afterwards) calls his girlfriend. Let $X$ be the time (in hours) until he calls his mother, and let $Y$ be the time (in hours) until he calls his girlfriend. Since he always calls his mother first, then $X < Y$. So let the joint density of the time be

$$f_{XY}(x,y) = \begin{cases} 10e^{-3x-2y} & \text{for } 0 < x < y, \\ 0 & \text{otherwise} \end{cases}$$

Given that $X = 0.5$, find the conditional probability that $Y > 2/3$. In other words, find $p(Y > \frac{2}{3} \mid X = 1/2)$. [20]

$$f_X(x) = \int_{y : x}^{\infty} 10e^{-3x-2y} \, dy = -5e^{-3x-2y} \bigg|_{y : x}^{\infty} = 5e^{-5x} ; \ x \geq 0$$

In particular, $f_X(1/2) = 5e^{-5/2}$.

Now the conditional density of $Y$ given $X = 1/2$ is

$$f_{Y/X}(y \mid x = 1/2) = \frac{f_{XY}(y, 1/2)}{f_X(1/2)} = \frac{10e^{-3(1/2)-2y}}{5e^{-5/2}} ; \ y > 1/2$$

$$= 2e^{1-2y} ; \ y > 1/2$$

$$P(Y > \frac{2}{3} \mid x = 1/2) = \int_{1/2}^{2/3} f_{Y/X}(y \mid x = 1/2) \, dy$$

$$= \int_{1/2}^{2/3} 2e^{1-2y} \, dy$$

$$= \left[ e^{1-2y} \right]_{1/2}^{2/3} = e^{-4/3} - e^{-1/3} \approx 0.71653$$
7. Consider the random variables $X$ is Alice's waiting time, and $Y$ is Bob's waiting time. Let $W = \max(X, Y)$, i.e., $W$ is either Alice's waiting time or Bob's waiting time, whichever is larger. The joint pdf of $X$ and $Y$ be

$$f_{XY}(x, y) = \begin{cases} 15e^{-3x-5y} & \text{for } x \geq 0, \ y \geq 0, \\ 0 & \text{otherwise} \end{cases}$$

Find $E(W)$. [15]

$$F_W(w) = P(W \leq w)$$
$$= P(\max(X, Y) \leq w)$$
$$= P(X \leq w, Y \leq w)$$
$$= \int_0^w \int_0^w 15e^{-3x-5y} \, dy \, dx$$
$$= \int_0^w -3e^{-3x-5y} \bigg|_{y=0}^{y=w} \, dx$$
$$= \int_0^w (-3e^{-3x-5w} + 3e^{-3x}) \, dx$$
$$= e^{-3x-5w} \bigg|_{x=0}^{x=w}$$
$$= e^{-8w} - e^{-5w} - e^{-3w} + 1$$

$$E(w) = \int_0^w -8we^{-8w} \, dw + \int_0^w 5we^{-5w} \, dw + \int_0^w 3we^{-3w} \, dw$$
$$= -8\left(\frac{12}{8^2}\right) + 5\left(\frac{12}{5^2}\right) + 3\left(\frac{12}{3^2}\right)$$
$$= -\frac{1}{8} + \frac{1}{5} + \frac{1}{3}$$
$$= \frac{49}{120} = 0.408$$

8. Suppose that $X$ and $Y$ are independent random variables with $\text{Var}(X) = 1$, $\text{Var}(Y) = 2$. Find $\text{Var}(1 - 2X + 3Y)$. [5]

$$\text{Var}(1 - 2X + 3Y) = 0 + (-2)^2 \text{Var}(X) + (3)^2 \text{Var}(Y)$$
$$= (4)(1) + (9)(2)$$
$$= 22$$
\[ \text{Cor}(x, y) = p = \frac{\text{Cov}(x, y)}{\text{S.d.}(x) \times \text{S.d.}(y)} = \frac{-2.4}{2.4} = -1 \]

9. Each day, Amy eats lunch at Wiley. She chooses pizza as her main dish with probability 40%, and her behavior each day is independent of all the other days. So if \( X \) denotes the number of days she chooses pizza in a 10-day period, then \( X \) is binomial with parameters \( n=10 \) and \( p=0.4 \). Let \( Y=10-X \) denote the number of days in which Amy does not eat pizza. Find the covariance \( \text{Cov}(X, Y) \) & \( \text{Cor}(X, Y) \) of \( X \) and \( Y \). [10+5]

\[ E(x) = n \times p = (10)(0.4) = 4 \]
\[ E(y) = E(10-x) = 10 - E(x) = 10 - 4 = 6 \]
\[ E(xy) = E((x(10-x)) = E(10x-x^2) = 10E(x) - E(x^2) \]

We know \( \text{var}(x) = E(x^2) - [E(x)]^2 = n \times p \times q = (10)(0.4)(0.6) = 2.4 \)

So \( E(x^2) = (2.4) + (4)^2 = 18.4 \)

Hence \( E(xy) = (10)(4) - 18.4 = 21.6 \)

Thus \( \text{Cov}(x, y) = E(xy) - E(x) \times E(y) \)
\[ = 21.6 - (4)(6) = -2.4 \]

10. Suppose that a treatment for back pain has three possible outcomes:
   O1 - Complete cure (no pain) – (30% chance)
   O2 - Reduced pain – (50% chance)
   O3 - No change – (20% chance)

Suppose the treatment is applied to \( n = 4 \) patients suffering back pain and let \( X \) = the number that result in a complete cure, \( Y \) = the number that result in just reduced pain, and \( Z \) = the number that result in no change. Find the probability of \( X + Y = 1 \). [5]

\[ P(X + Y = 1) = P(X = 1, Y = 0) + P(X = 0, Y = 1) \]
\[ = \frac{4!}{0!1!1!3!} (0.3)^1 (0.5)^0 (0.2)^3 \]
\[ + \frac{4!}{0!1!1!3!} (0.3)^0 (0.5)^1 (0.2)^3 \]
\[ = 0.0256 \]