MATH 321  Exam #2  100 Points  Oct 26, 2012

You must show enough work, or give sufficient explanation, in each problem to clearly indicate how you obtain your answer. No credit will be given for a problem if there is insufficient work/explanation.

1. Suppose a particular state allows individuals filing tax returns to itemize deductions only if the total of all itemized deductions is at least $5000. Let $X$ (in 1000's of dollars) is the total of itemized deductions on a randomly chosen form. Assume that $X$ has a the pdf

$$f(x, \alpha) = \begin{cases} \frac{k}{x^\alpha} & x \geq 5 \\ 0 & \text{otherwise} \end{cases}$$

a. Find the value of $K$. What restrictions on $\alpha$ is necessary? [4+1]

$$1 = k \int_{5}^{\infty} x^{-\alpha} \, dx = k \left[ \frac{x^{-\alpha + 1}}{-\alpha + 1} \right]_{5}^{\infty} = k \left[ \frac{1}{(1-\alpha) \cdot 5^{\alpha - 1}} \right] = k \left[ 0 - \frac{1}{(1-\alpha) \cdot 5^{\alpha - 1}} \right] = 1$$

$$\Rightarrow k = \frac{1}{(\alpha - 1) \cdot 5^{\alpha - 1}} \quad \text{provided } \alpha > 1$$

b. What is the expected total deduction on a randomly chosen form? [5]

$$E(X) = \int_{5}^{\infty} x \cdot \frac{k}{x^\alpha} \, dx = k \int_{5}^{\infty} x^{-\alpha + 1} \, dx = k \left[ \frac{x^{1-\alpha + 1}}{1-\alpha + 1} \right]_{5}^{\infty} = k \left[ \frac{1}{\cdot 5^{\alpha - 2}} \right]$$

$$= \frac{5^{\alpha - 1}}{(\alpha - 1)} = \frac{5 \cdot 5^{\alpha - 2}}{(\alpha - 2)}$$

$$= k \left[ 0 + \frac{1}{\cdot 5^{\alpha - 2}} \right]$$

$$= \frac{5^{\alpha - 1}}{(\alpha - 1)}$$

(c) Show that $Y = \ln\left( \frac{X}{5} \right)$ has an exponential distribution with the parameter $(\alpha - 1)$. [10]

$$P(Y \leq y) = P\left( \ln\left( \frac{X}{5} \right) \leq y \right) = P\left( \frac{X}{5} \leq e^y \right) = P\left( X \leq 5e^y \right)$$

$$= 1 - \left( \frac{5}{5e^y} \right)^{\alpha - 1} = 1 - e^{-y(\alpha - 1)}$$

$1 - e^{-y(\alpha - 1)}$, the cdf of an exponential r.v. with parameter $(\alpha - 1)$
2. Consider the following probability density function:

\[
f_X(x) = \begin{cases} 
  kx & 0 \leq x < 2 \\
  2k & 2 \leq x < 4 \\
  k(6-x) & 4 \leq x < 6 \\
  0 & \text{otherwise}
\end{cases}
\]

Compute the cumulative distribution function \( F_X(x) \). [5]

\[
F_X(x) = \begin{cases} 
  \int_0^x ky \, dy & 0 \leq x < 2 \\
  0 & 0 \leq x \leq 2 \\
  \int_0^x ky \, dy + \int_2^x 2k \, dy & 2 \leq x < 4 \\
  \int_0^x ky \, dy + \int_2^x 2k \, dy + \int_4^x (6-y) \, dy & 4 \leq x < 6 \\
  0 & x \geq 6
\end{cases}
\]

3. Two species are competing in a region for control of a limited amount of a certain resource. Let \( X \) = the proportion of the resource controlled by species 1 and suppose \( X \) has pdf

\[
f(x) = \begin{cases} 
  1 & 0 \leq x \leq 1 \\
  0 & \text{otherwise}
\end{cases}
\]

Find the \( E[h(X)] \), where \( h(X) = \max(X, 1-X) \). [5+5]

[Hint: Split the interval \([0, 1]\) and find \( h(X) \) in that interval]

\[
E(h(X)) = \int_0^1 \max(1, 1-x) \times 1 \, dx
\]

\[
= \int_0^{1/2} (1-x) \, dx + \int_0^{1/2} x \, dx
\]

\[
= \left[ x - \frac{x^2}{2} \right]_0^{1/2} + \left[ \frac{x^2}{2} \right]_0^{1/2}
\]

\[
= \left( \frac{1}{2} - \frac{1}{8} \right) - \left( \frac{1}{8} - \frac{1}{16} \right)
\]

\[
= 1 - \frac{1}{4}
\]

4. You take your laundry to a Laundromat and pay a dollar each time you run the dryer. After much study you have determined that the length of time the dryer runs on one dollar is a continuous uniform random variable bounded by 6 and 12 minutes. If a pile of your wet laundry needs 8 minutes to dry, what is the probability that you will need to pay two dollars to have the dryer run long enough to finish the load? [5]

If the dryer runs less than 8 minutes for the first dollar, you need to pay a second dollar

\[
P(\$2 \, \text{required}) = P(X < 8) = (8-6)(\frac{1}{12-6}) = \frac{1}{3} = 0.3333\bar{3}
\]
5. The peak temperature $T$, in degrees Fahrenheit, on a July day in Antarctica is a Normal (Gaussian) random variable with a variance of 225. With probability 0.5, the temperature $T$ exceeds 10 degrees. What is $P[T > 32]$, the probability the temperature is above freezing? [5+5]

\[ P(T > 10) = 1 - P(T \leq 10) = 1 - \Phi \left( \frac{10 - \mu}{15} \right) = \frac{1}{2} \]

\[ = \Phi \left( \frac{10 - \mu}{15} \right) = \frac{1}{2} \Rightarrow \mu = 10 \]

\[ P(T > 32) = 1 - P(T \leq 32) \]

\[ = 1 - \Phi \left( \frac{32 - 10}{15} \right) \]

\[ = 1 - \Phi (1.47) \]

\[ = 1 - 0.929 = 0.071 \]

6. Suppose only 75% of all drivers in a certain state regularly wear seat belt. A random sample of 500 drivers is selected. What is the probability that fewer than 400 of those in the sample regularly wear a seat belt? [10]

\[ n_p = 0.75 \]

\[ n = n_p \cdot 500 = 375 \]

\[ \sigma = \sqrt{n_pq} = 9.68 \]

\[ X \sim \text{Bin}(500, 0.75) \]

\[ = N(375, 9.68) \]

\[ P(X < 400) = P(Z \leq 2.53) \]

\[ = \Phi (2.53) \]

\[ = 0.9943 \]

7. In Tim’s new house, the water heater is actually quite old. Its remaining lifetime $X$ follows the pdf $f_X(x) = 4e^{-4x}$ for $x > 0$ in years.
a). What is the probability that his water heater lasts 6 months? [5]

\[ 6 \text{ months} = 0.5 \text{ years} \]

\[ X \sim \text{Exp}(\lambda) \Rightarrow f_X(x) = P(X \leq x) = 1 - e^{-\lambda x} \quad ; \quad x > 0 \]

We want $P(X \leq 0.5) = f(0.5) = 1 - e^{-4(0.5)} = 0.865$

\[ P(X > 0.5) = 1 - P(X \leq 0.5) \]

\[ = 1 - 0.865 = 0.135 \]
b). Given that his water heater lasts for 6 months, what is the probability that it lasts for 4 more months after that? [5]

\[
P(\frac{x}{12} \geq \frac{10}{12} / x > \frac{6}{12}) = P(x \geq \frac{4}{12})
\]

(By memory less property)

\[
= e^{-4 \cdot \frac{4}{12}}
\]

\[
= 0.2636
\]

8. Suppose that when a transistor of a certain type is subjected to an accelerated life test, the lifetime \(X\) (in weeks) has a gamma distribution with mean of 40 and variance of 320.

a). What is the probability that a transistor will last between 1 and 40 weeks? [5]

\[
\alpha = \mu = 40 \\
\alpha \beta^2 = \sigma^2 = 320
\]

\[
P(1 \leq x \leq 40) = F^x(\frac{40}{8}, 5) - F^x(\frac{1}{8}, 5)
\]

\[
= F^x(5, 5) - F^x(\frac{1}{8}, 5)
\]

\[
= 0.560 - 0
\]

\[
= 0.560
\]

b). What is the probability that a transistor will last at most 40 weeks? Is the median of the lifetime distribution less than 40? Why or why not? [5]

\[
P(x \leq 40) = F^x(\frac{40}{8}, 5)
\]

\[
= F^x(5, 5) = 0.560
\]

Yes,
9. The time between severe earthquakes at a given region follows a log-normal distribution with a coefficient of variation of 40% \( (C.V. = \frac{\sigma}{\mu} \times 100) \). The expected time between severe earthquakes is 80 years (\( \mu \)).

(a) Determine the parameters of this log-normally distributed interval time \( T \). [5]

\[
\begin{align*}
\mathbb{E}(X) &= \mu_T = e^\mu + \frac{1}{2} \sigma^2 = 80 \\
\mathbb{V}(X) &= \sigma_T^2 = e^{2\mu} + \sigma^2 \left( e^{\sigma^2} - 1 \right) \\
\mu &= \ln(\mathbb{E}(X)) - \frac{1}{2} \ln \left( 1 + \frac{\mathbb{V}(X)}{\mathbb{E}(X)^2} \right) \\
\sigma^2 &= \ln \left( 1 + \frac{\mathbb{V}(X)}{\mathbb{E}(X)^2} \right)
\end{align*}
\]

(b) Determine the probability that a severe earthquake will occur within 20 years from the previous one. [5]

10. Suppose that the Moment Generating Function for \( X \) is \( M_X(t) = \frac{2e^t}{7-5e^t} \). Determine the mean and variance for \( X \). [10]

\[
M_X(t) = \frac{2e^t}{7-5e^t} = \frac{2/7 e^t}{1 - 5/7} \quad \text{which is M.G.F. Geometric Random Variable with } \mathbb{E}(X) = \frac{1}{p} \quad \text{and } \mathbb{V}(X) = 2/p^2 \quad \left[ p = \frac{7}{2} \right]
\]

\[
\begin{align*}
M_X' (t) &= \frac{d}{dt} \left( \frac{2e^t}{7-5e^t} \right) = \frac{(7-5e^t)(2e^t) - (2e^t)(-5e^t)}{(7-5e^t)^2} \\
&= \frac{7}{2} = 3.5 \\
M_X'' (t) &= \frac{d}{dt} \left( \frac{14e^t}{(7-5e^t)^2} \right) = \frac{(7-5e^t)^2 (14e^t) - (14e^t)(2(7-5e^t))}{(7-5e^t)^4} \\
&= \frac{(2^2)(14) - 20}{16} = \frac{336}{16} = 21
\end{align*}
\]

\[ \mathbb{E}(X^2) = 21 \]