MATH 321B  Exam #2  100 Points Total  Oct 21, 2011

You must show enough work, or give sufficient explanation, in each problem to clearly indicate how you obtain your answer. No credit will be given for a problem if there is insufficient work/explanation.

1. Suppose that $X$ is a continuous random variable with p.d.f.
   
   \[ f_X(x) = x + c \text{ for } 0 < x < c. \]

   Find $c$. [5]

   \[
   \int_{0}^{c} (x + c) \, dx = 1 \\
   = \frac{x^2}{2} + cx \bigg|_{0}^{c} = 1 \quad \Rightarrow \quad \frac{c^2}{2} + c^2 = 1 \quad \Rightarrow \quad \frac{3c^2}{2} = 1 \quad \Rightarrow \quad c = \sqrt{\frac{2}{3}}.
   \]

2. You arrive at a bus stop at 10.00, knowing that the bus will arrive at some time uniformly distributed between 10.00 and 10.30. If at 10.15 the bus has not yet arrived, what is the probability that you will have to wait at most an additional 10 minutes?

   Note: If $Y$ denotes the time that you wait at the stop until the bus arrives, then $Y$ has a $U(0, 30)$ distribution and the problem is asking for $P(Y \leq 25 \mid Y > 15)$. [5]

   \[ f_Y(y) = \begin{cases} \frac{1}{30}, & 0 < y < 30 \\ 0, & \text{otherwise} \end{cases} \]

   \[ F_Y(y) = P(Y \leq y) = \int_{0}^{y} \frac{1}{30} \, dx = \frac{y}{30}, \quad 0 \leq y \leq 30. \]

   \[ P(15 < Y \leq 25) = \int_{15}^{25} \frac{1}{30} \, dx = \frac{x}{30} \bigg|_{15}^{25} = \frac{1}{3}. \]

   \[ P(Y > 15) = \int_{15}^{30} \frac{1}{30} \, dx = \frac{x}{30} \bigg|_{15}^{30} = \frac{1}{2}. \]

   \[ P(Y \leq 25 \mid Y > 15) = \frac{P(15 < Y \leq 25)}{P(Y > 15)} = \frac{\int_{15}^{25} \frac{1}{30} \, dx}{\int_{15}^{30} \frac{1}{30} \, dx} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}. \]

3. Let $Y$ be a random variable with probability density function

   \[ f_Y(y) = \begin{cases} \frac{1}{2}, & -1 \leq y \leq 0 \\ y, & 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases} \]
a. Graph the function \( f(y) \) and find the probability of \( P(-1/2 \leq y \leq 1/3) \).

\[
P(-\frac{1}{2} \leq y \leq \frac{1}{3}) = \int_{-\frac{1}{2}}^{\frac{1}{3}} f(y) \, dy
\]

\[
= \int_{-\frac{1}{2}}^{0} f(y) \, dy + \int_{0}^{\frac{1}{3}} f(y) \, dy
\]

\[
= \frac{1}{4} + \frac{1}{18} = \frac{11}{36} \approx 0.3056.
\]

b. Find the mean of \( Y \).

\[
E(Y) = \int_{\infty}^{\infty} y f(y) \, dy
\]

\[
= \int_{-\frac{1}{2}}^{0} y f(y) \, dy + \int_{0}^{\frac{1}{3}} y^2 f(y) \, dy
\]

\[
= \frac{1}{4} - \frac{1}{6} - \frac{1}{3} = -\frac{1}{12} = -0.0833.
\]

c. Find the distribution function, \( F_Y(y) \), of \( Y \).

\[
F_Y(y) = P(Y \leq y)
\]

For \( 0 \leq y \leq 1 \):

\[
F_Y(y) = \int_{-\frac{1}{2}}^{y} f(y) \, dy = \frac{1}{2} + \frac{y^2}{2}
\]

For \( y \leq -1 \):

\[
F_Y(y) = 0
\]

For \(-1 < y < 0 \):

\[
F_Y(y) = \int_{-\frac{1}{2}}^{y} f(y) \, dy = \frac{1}{2} y + \frac{1}{2}
\]

For \( y \geq 1 \):

\[
F_Y(y) = 1
\]

4. Suppose that the distance (in hundreds of miles) driven by a trucker in one day is a continuous random variable \( D \) whose probability density function (pdf) is given by:

\[
g(d) = \begin{cases} 
\frac{d}{32} & 0 < d < 8 \\
0 & \text{otherwise}
\end{cases}
\]

Let the random variable \( T \) represent the time (in number of days) required for the trucker to make a 3000-mile cross-country trip, so \( T = \frac{3000}{d} \). Determine the expected value of \( T \).

\[
E(T) = E(\frac{3000}{d}) = \int_{0}^{8} \frac{3000}{d} g(d) \, dd = \int_{0}^{8} \frac{3000}{d} \cdot \frac{d}{32} \, dd
\]

\[
= \frac{3000}{32} \int_{0}^{8} \frac{1}{d} \, dd = \frac{2400}{32} = 75 \text{ days}.
\]
5. Suppose we draw a sample of 1,500 Americans and want to assess whether the representation of blacks in the sample is accurate. We know that about 12% of Americans are black, so we expect $X$, the number of blacks in the sample, to be around 180. Allowing a little leeway, what is the probability that the sample contains 170 or fewer blacks? [10]

$$n = 1500$$
$$p = 0.12$$

$$n \cdot p = (1500)(0.12) = 180 > 5$$
$$n \cdot q = (1500)(0.88) = 1320 > 5$$

Use normal approximation to binomial

$$P(X \leq 170) = P\left( Z \leq \frac{170 + 0.5 - 180}{12.5857} \right) = \Phi (-0.75) = 0.2266$$

6. Suppose that the lifetime of an electrical component is Exponential with parameter $\lambda = 2$ per year. Further suppose that the component is known to be functioning one year after it is put in use. Find the probability that it fails before the 2-year mark. [5]

$$P(X < 2 / X > 1) = \frac{P(1 \leq X \leq 2)}{P(X > 1)} = \frac{F(2) - F(1)}{1 - F(1)} = \frac{(1 - e^{-2(2)}) - (1 - e^{-2(1)})}{e^{-2(1)}}$$

Using lack of memory property:

$$P(X < 2 / X > 1) = P(X < 1) = 1 - e^{-2(1)} = 1 - 0.1353 = 0.8647$$

7. The time required to repair a machine has an exponential distribution with mean 2 hours.

a. What is the probability that a repair time exceeds 2 hours? [5]

$$P(X > 2) = 1 - P(X \leq 2)$$

$$= 1 - \left[ 1 - e^{-\frac{1}{2}} \right] = e^{-1} = 0.368$$

b. If five of these machines need repair, what is the probability that at most one of them has a repair time that exceeds 2 hours? [5]

$$n = 5$$
$$p = 0.368$$

$$P(X \leq 1) = P(X = 1) + P(X = 0)$$

$$= (5) (0.368) (0.632)^4 + (\frac{5}{6}) (0.368) (0.632)^5$$

$$= 0.394$$
8. The number of calls that arrive at a help desk during a one-hour period has a Poisson distribution with a mean of 10. You have a half-hour shift at the help desk.

a. Let $N$ be the number of calls that arrive during your shift. Find $P(N \geq 3)$. \[ \lambda = 10 \times \frac{1}{2} = 5 \]

\[
P(N \geq 3) = 1 - P(N \leq 2)
= 1 - \left[ P(N = 0) + P(N = 1) + P(N = 2) \right]
= 1 - \left[ \frac{e^{-5}5^0}{0!} + \frac{e^{-5}5^1}{1!} + \frac{e^{-5}5^2}{2!} \right]
= 1 - \left[ 0.0067 + 0.0337 + 0.0842 \right]
= 0.8754
\]

b. Let $W$ be the time in hours between the arrival of your first call and the arrival of your second call follow exponential distribution with parameter $\lambda = 10$. Find $p(W \leq 0.1)$? \[ \lambda = 10 \]

\[
P(W \leq 0.1) = F(W) = 1 - e^{-(10)(0.1)} = 0.632
\]

\[
P(W \leq 0.1) = \int_0^{0.1} \lambda e^{-\lambda x} dx = 10 \int_0^{0.1} e^{-10x} dx = 0.632
\]

c. Let $Y$ be the total time in hours elapsed between the start of your shift and the arrival of your second call. Find $P(Y < 0.5)$ \[
\alpha = 2
\beta = \frac{1}{\lambda} = \frac{1}{10}
\]

\[
F(0.5, 2, \beta = \frac{1}{10}) = F^\alpha(0.5, 2) = 0.960
\]

\[
P(Y < 0.5) = \int_0^{0.5} (10y) e^{-10y} dy
= 100 \left[ -\frac{1}{10} y e^{-10y} \right]_0^{0.5} + \int_0^{0.5} \frac{1}{10} e^{-10y} dy
= 100 \left[ -\frac{1}{10} \left( -\frac{1}{100} e^{-5} \right) + \frac{1}{100} \right]
= 0.960
\]

9. Let $X$ = hourly median power (in decibels) of received radio signals transmitted between two cities. It is believed that the lognormal distribution provides a reasonable probability model for $X$. If the parameter values are $\mu = 3.5$ and $\sigma = 1.2$, (i) calculate the probability that received power is between 50 and 250 dB \[
P(50 \leq x \leq 250) = P(x \leq 250) - P(x \leq 50)
= P\left( z \leq \frac{\ln(250) - 3.5}{1.2} \right) - P\left( z \leq \frac{\ln(50) - 3.5}{1.2} \right)
= \Phi(1.68) - \Phi(0.34)
= 0.9545 - 0.6331
= 0.3214
\]
\[ E(X) = e^{\mu + \sigma^2/2} \]

\[ = e^{3.5 + (1.2)^2/2} = e^{4.22} = 68.033 \]

(ii) Find the probability that \( X \) is less than its mean value? [5]

\[ P(X \leq E(X)) = P(X \leq 68.033) \]

\[ = P\left( Z \leq \frac{\ln(68.033) - 3.5}{1.2} \right) \]

\[ = \Phi(0.6) \]

\[ = 0.7257 \]

10. An individual's credit score is a number calculated based on that person's credit history which helps a lender determine how much he/she should be loaned or what credit limit should be established for a credit card. An article in the Los Angeles Times gave data which suggested that a beta distribution with parameters \( \alpha = 150; \beta = 850; \alpha = 8; \beta = 2 \) would provide a reasonable approximation to the distribution of American credit scores. [Note: credit scores are integer-valued].

What is the approximate probability that a randomly selected score will exceed 750 (which lenders consider a very good score)? [5]
11. The random variable \(X\) has an exponential random variable with mean of 1. The random variable \(Y\) is defined to be \(Y = 2 \ln(X)\). Find the p.d.f of \(Y\) and verify that. \([5]\)

\[
\begin{align*}
F_Y(y) &= P(Y \leq y) \\
&= P(2 \ln(X) \leq y) \\
&= P(\ln(X^2) \leq y) \\
&= P(X^2 \leq e^y) \\
&= P(X \leq e^{y/2}) \\
&= 1 - e^{-y/2} \\
\end{align*}
\]

\[
\begin{align*}
f_Y(y) &= F_Y'(y) = \frac{1}{2} e^{y/2} e^{-y/2} \\
&= \frac{1}{2} e^{y/2} \\
&= e^{-y/2} \cdot \frac{1}{2} e^{y/2} \quad -\infty < y < \infty
\end{align*}
\]

12. A random variable \(Y\) has moment-generating function

\[M_X(t) = (\frac{1}{3} e^t + \frac{2}{3})^5\]

Find the mean and variance of \(X\). \([5]\)

\[
\begin{align*}
E(X) &= M_X'(0) = 5 \left(\frac{1}{3} e^t + \frac{2}{3}\right)^4 \cdot \frac{1}{3} e^t \\
M_X(t) &= \left[\frac{5}{3} e^t \left(\frac{1}{3} e^t + \frac{2}{3}\right)^4\right] \\
M_X'(t) &= \frac{d}{dt} \left[\frac{5}{3} e^t \left(\frac{1}{3} e^t + \frac{2}{3}\right)^4\right] \\
&= \frac{5}{3} e^t \left(\frac{1}{3} e^t + \frac{2}{3}\right)^4 + \frac{20}{3} e^t \left(\frac{1}{3} e^t + \frac{2}{3}\right)^3 \cdot \frac{1}{3} e^t \\
M_X'(t)_{|t=0} &= \frac{5}{3} \cdot (1) + \frac{20}{3} \cdot (1) = \frac{35}{9} = M_2 = E(X^2) \\
V(X) &= E(X^2) - \left[E(X)\right]^2 = \frac{35}{9} - \left(\frac{5}{3}\right)^2 = \frac{10}{9}
\end{align*}
\]