1. Suppose that the number of accidents at the Iowa power plant follows the Poisson process with
the average rate of 0.40 accidents per week. Assume all weeks are independent of each other.
a) Find the probability that exactly 4 accidents will occur in two months (8 weeks). [5]

\[ \lambda = 0.4 \text{/week} \Rightarrow 0.4 \times 8 = 3.2 / 2 \text{ months} \Rightarrow \lambda t = 3.2 \]

\[ P(X = 4) = \frac{e^{-3.2} (3.2)^4}{4!} = 0.178 \]

b) Find the probability that there will be 5 accident-free weeks in two months (8 weeks). [5+5]

\[ P(X = 0) = \frac{e^{-0.4} (0.4)^0}{0!} = 0.6703 \]

\[ X \sim B(n = 8, p = 0.6703) \]

\[ P(X = 5) = \binom{8}{5} (0.6703)^5 (1 - 0.6703)^3 \]

\[ = 0.2716. \]

2. A recruiter is interviewing job candidates. From past data, the recruiter thinks that about 30% of
the potential job candidates have the qualifications necessary to be hired for a certain middle
management position. Assume independence. Find the probability that the third suitable job
prospect is the eighth candidate interviewed. [5]

\[ X \sim \text{NB}(\gamma = 3, \ p = 0.3) \]

\[ P(X = 5) = \binom{5 + 3 - 1}{3 - 1} (0.3)^3 (1 - 0.3)^5 \]

\[ = 0.0953. \]
3. There are five bananas and seven oranges in the refrigerator. Four fruits are chosen at random to serve guest. What is the probability that exactly two of the fruits will be oranges? And what is the expected number of oranges chosen? [3+2]

\[ X \sim \text{HY}(\eta = 4, M = 7, N = 12) \]

\[ P(X = 2) = \frac{7}{2} \times \frac{12 - 7}{4 - 2} \times \frac{12}{4} = 0.4242 \]

\[ E(X) = \eta \times \frac{M}{N} = 4 \times \frac{7}{12} = 2.333 \]

4. A simple model for describing mortality in the general population in a particular country is given by the probability density function

\[ f(y) = \begin{cases} 252y^6(1-y)^2 & ; 0 < y < 1 \\ 0 & ; \text{otherwise} \end{cases} \]

\[ f(y) = \frac{252}{10^{18}} y^6 (100-y)^2 \]

\[ 0 \leq y \leq 100 \]

a) Verify that \( f(y) \) is a valid probability density function. [5]

\[ \int_{-\infty}^{\infty} f(y) \, dy = \int_{0}^{1} 252y^6(1-y)^2 \, dy = 1 \]

\[ = 252 \int_{0}^{1} (y^6 - 2y^7 + y^8) \, dy = 1 \]

\[ = 252 \left[ \frac{y^7}{7} - \frac{2y^8}{8} + \frac{y^9}{9} \right]_0^1 = 252 \left( \frac{1}{252} \right) = 1 \]

b) Based on this model, which event is more likely [5 + 5]

A: A person dies between the ages of 70 and 80

OR

B: A person lives past age 80?

\[ 252 \left[ \frac{y^7}{7} - \frac{2y^8}{8} + \frac{y^9}{9} \right]_0^{0.8} \]

\[ = 252 \left[ \frac{0.8^7}{7} - \frac{2 \times 0.8^8}{8} + \frac{0.8^9}{9} \right] \]

\[ = 0.275 \]

\[ 252 \left[ \frac{y^7}{7} - \frac{2y^8}{8} + \frac{y^9}{9} \right]_0^{0.7} \]

\[ = 252 \left[ \frac{0.7^7}{7} - \frac{2 \times 0.7^8}{8} + \frac{0.7^9}{9} \right] \]

\[ = 0.2618 \]
c) Given that a randomly selected individual just celebrated his 60th birthday, find the probability that he will live past age 80. \[ p(Y > 0.8 / Y > 0.6) = \frac{p(Y > 0.8 \cap Y > 0.6)}{p(Y > 0.6)} = \frac{p(Y > 0.8)}{p(Y > 0.6)} \]

\[ \approx \frac{0.2618 \text{ [from b]}}{0.768212} = \frac{0.2618}{0.768212} \]

d) Find the value of \( y \) that maximizes \( f(y) \) (mode). \[ p(y) = 252 y^6 (1 - y)^2 \]

\[ \ln p(y) = \ln(252) + 6 \ln(y) + 2 \ln(1 - y) \]

\[ \frac{d \ln(p(y))}{dy} = 0 \implies \frac{6}{y} + \frac{2}{1 - y} (-1) = 0 \]

\[ \implies \frac{6}{y} - \frac{2}{1 - y} = 0 \implies 2y = 6 - 6y \implies 8y = 6 \implies y = \frac{6}{8} = \frac{3}{4} \approx 0.75 \]

e) Find the (average) life expectancy. \[ E(Y) = 252 \int_0^1 y (y^6 - 2y^7 + y^8) dy \]

\[ = 252 \int_0^1 (y^7 - 2y^8 + y^9) dy \]

\[ = 252 \left[ \frac{y^8}{8} - \frac{2y^9}{9} + \frac{y^{10}}{10} \right]_0^1 \]

\[ = 252 \left( \frac{1}{360} \right) = 0.7. \]
5. Suppose that the wrapper of a certain candy bar lists its weight as 2.13 ounces. Suppose that the actual weights of these candy bars vary according to a normal distribution with mean \( \mu = 2.2 \) ounces and standard deviation \( \sigma = 0.06 \) ounces. In a random sample of 5 candy bars, what is the probability that at least one of the candy bars weighs less than the advertised weight? [5+5]

\[
P(x < 2.13) = P\left( z < \frac{2.13 - 2.2}{0.06} \right) = P(z < -1.17) = \Phi(-1.17) = 0.1210.
\]

\( X \sim b(n = 5, p = 0.1210) \)

\[
P(x \geq 1) = 1 - P(x < 1) = 1 - P(x = 0) = 1 - \left( \binom{5}{0} (0.1210)^0 (1 - 0.1210)^5 \right) = 0.97525
\]

6. Suppose that the average length of time spent eating lunch in a particular restaurant is 0.6 hours. Suppose further that the time spent eating lunch is exponentially distributed.

(a). What is the probability that a given individual will spend less than 2 hours eating lunch? [5]

\[
E(x) = 0.6 = \frac{1}{\lambda} \Rightarrow \lambda = \frac{1}{0.6} = \frac{10}{6}
\]

\[
P(x < 2) = \frac{10}{6} \int_0^2 e^{-\frac{10}{6} x} dx = \frac{10}{6} \left[ -\frac{10}{6} e^{-\frac{10}{6} x} \right]_0^2 = -e^{\frac{20}{3}} + 1
\]

\( \text{(or)} \)

\[
F_X(x) = P(x \leq x) = 1 - e^{\lambda x} ; x = 2, \lambda = \frac{10}{6}
\]

(b). If you are waiting for a particular person to finish eating so that you can have his table, and the waitress tells you that he has already been eating for a half an hour. What is the probability that the table will still be occupied after another hour? [5]

\[
P(x > 1.5/ x > 0.5) = P(x > 1) \quad \text{by lack of memory property}
\]

\[
P(x > 1) = 1 - P(x \leq 1) = 1 - (e^{-\lambda x}) = 1 - (1 - e^{-\lambda x})
\]

\[
e^{-\lambda x} \quad \text{where} \quad \lambda = \frac{1}{0.6} = \frac{10}{6}
\]

\[
= e^{\left(-\frac{10}{6}\right)} \quad x = 1
\]
7. Suppose the survival time $X$ in weeks of a randomly selected male mouse exposed to 240 rads of gamma radiation has a gamma distribution with $\alpha = 8$ and $\beta = 15$. What is the probability that a mouse survives between 60 and 120 weeks? [5]

$$X \sim \text{Gamma} (\alpha = 8, \beta = 15)$$

$$P \left( 60 \leq X \leq 120 \right) = F^* \left( \frac{120}{15}, 8 \right) - F^* \left( \frac{60}{15}, 8 \right)$$

$$= F^* \left( 8, 8 \right) - F^* \left( 4, 8 \right)$$

$$= 0.547 - 0.051$$

$$= 0.496$$  

(From Table IV)