MATH 321  Exam #1  100 Points  Sep 27, 2013

You must show enough work, or give sufficient explanation, in each problem to clearly indicate how you obtain your answer. No credit will be given for a problem if there is insufficient work/explanation.

1. A point is chosen at random inside the following quadrilateral (the boundary is allowed too). What is the sample space? [5]

   The sample space is
   \[ S = \{ (x,y) \mid 0 \leq x \leq 2, \ 0 \leq y \leq 4-x \} \]

2. Four married couples have bought 8 seats in a row for a concert. In how many different ways can they be seated?

   1. If each couple is to sit together? [5]

      \[
      (8 \cdot 1) \cdot (6 \cdot 1) \cdot (4 \cdot 1) \cdot (2 \cdot 1) = 384 \text{ ways.}
      \]
      
      (OR) \( 4! \cdot (2! \cdot 2! \cdot 2!) = 384 \)

   2. If all the men sit together to the right of all the women? [5]

      \[ 4! \cdot 4! = 576 \text{ ways.} \]

3. Suppose each day that you drive to work a traffic light that you encounter is either green with probability 7/16, red with probability 7/16, or yellow with probability 1/8, independent of the status of the light on any other day. If over the course of five days, G, Y, and R denote the number of times the light is found to be green, yellow, or red, respectively, what is the probability that \( P[G = 2, Y = 1, R = 2] \)? [5]

   \[
   P(G = 2, Y = 1, R = 2) = \frac{5!}{2! \cdot 1! \cdot 2!} \left( \frac{7}{16} \right)^2 \left( \frac{1}{8} \right) \left( \frac{7}{16} \right)^2
   \]
4. Deer ticks can carry both Lyme disease and human granulocytic ehrlichiosis (HGE). In a study of ticks in the Midwest, it was found that 16% carried Lyme disease, 10% had HGE, and that 10% of the deer ticks that had either Lyme disease or HGE carried both diseases.

\[ (a) \text{ What is the probability } P[LH] \text{ that a tick carries both Lyme disease (L) and HGE (H)?} \ [5] \]

\[
P(L) = 0.16 \\
P(H) = 0.1 \\
P(LH/L) = 0.1
\]

Then \[ P(LNH) = (0.10) \cdot P(LH) = (0.10)[P(L) + P(H) - P(LH)] \]

\[ \Rightarrow P(LNH) = (0.10)\frac{(0.16 + 0.1)}{1.1} = 0.0236 \]

(b) What is the conditional probability that a tick has HGE given that it has Lyme disease? [5]

\[ P(H/L) = \frac{P(HL)}{P(L)} = \frac{0.0236}{0.16} = 0.1475 \]

5. The Review editor for a certain scientific journal decides whether the review for any particular book should be short (1–2 pages), medium (3–4 pages), or long (5–6 pages). Data on recent reviews indicates that 60% of them are short, 30% are medium, and the other 10% are long. Reviews are submitted in either Word or LaTeX. For short reviews, 80% are in Word, whereas 50% of medium reviews are in Word and 30% of long reviews are in Word. Suppose a recent review is randomly selected.

a. What is the probability that the selected review was submitted in Word format? [5]

\[ \text{Given:} \]

\[
P(S) = 0.6 \\
P(M) = 0.3 \\
P(L) = 0.1 \\
P(W/S) = 0.8 \\
P(W/M) = 0.5 \\
P(W/L) = 0.3
\]

By Law as Total Probability

\[ P(W) = P(W/S) \cdot P(S) + P(W/M) \cdot P(M) + P(W/L) \cdot P(L) \]

\[ = (0.8) \cdot (0.6) + (0.5)(0.3) + (0.3)(0.1) \]

\[ = 0.66 \]

b. If the selected review was submitted in Word format, what are the posterior probabilities of it being short? [5]

\[ \text{By the Bayes Theorem} \]

\[ P(S/W) = \frac{P(SW)}{P(W)} = \frac{P(SW) \cdot P(S)}{P(W)} = \frac{(0.8)(0.6)}{0.66} = 0.727 \]
6. In a gambling game a person draws a single card from an ordinary deck of 52 playing cards. A person is paid $15 for drawing a jack or a queen and $5 for drawing a king or an ace. A person who draws any other card pays $4. For a person who plays this game, denote by \( X \) his or her gain.

(a) Determine the probability mass function (pmf) of \( X \). [5]

\[
P(X = x) = \begin{cases} \frac{8}{52} = \frac{2}{13} & x = 15 \\ \frac{8}{52} = \frac{2}{13} & x = 5 \\ \frac{3}{52} = \frac{9}{13} & x = -4 \end{cases}
\]

(b) What is the expected gain \( E(X) \)? [5]

\[
E(X) = (-4)(\frac{9}{13}) + (5)(\frac{2}{13}) + (15)(\frac{3}{13}) = \frac{60 - 36 + 45}{13} = \frac{79}{13} \approx 6.08
\]

7. A certain electronic system contains 10 components. Suppose that the probability that each individual component will fail is 0.2 and that the components fail independently of each other. Given that at least one of the components has failed, what is the probability that at least two of the components have failed? [5]

\[
P(X \geq 2 / x \geq 1) = \frac{\frac{X(2 \text{and } x \geq 1)}{P(x \geq 1)}}{P(x \geq 1) - \frac{P(X < 2)}{P(X > 1)}} = \frac{1 - P(X < 2)}{1 - P(X < 1)}
\]

8. Anytime a child throws a Frisbee, the child’s dog catches the Frisbee with probability \( p \), independent of whether the Frisbee is caught on any previous throw. When the dog catches the Frisbee, it runs away with the Frisbee, never to be seen again. The child continues to throw the Frisbee until the dog catches it. Let \( X \) denotes the number of times the Frisbee is thrown.

(a) What is the probability distribution or PMF \( P_X(x) \)? [5]

\[
P_X(x) = \begin{cases} (1 - p)^{x-1} p & x = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}
\]

(b) If \( p = 0.2 \), what is the probability that the child will throw the Frisbee more than four times? [5]

\[
P(X > 4) = 1 - P(X \leq 4) = 1 - \left[ P(X = 4) + P(X = 3) + P(X = 2) + P(X = 1) \right] = (1 - p)^4 \text{ using the result } (0.8)^4 = 0.41 \text{ proved in class.}
\]

(c) If \( h(x) = 5X + 5 \), find \( E(h(x)) \) and \( V(h(x)) \). [5]

\[
E(5X + 5) = 5E(X) + 5 = 5(5) + 5 = 30
\]

\[
V(5X + 5) = 5^2 V(X) = 25 \left( \frac{0.8}{0.2^2} \right) = 500
\]
9. Suppose that a machine shop orders 500 bolts from a supplier. To determine whether to accept the shipment of bolts, the manager of the facility randomly selects 12 bolts. If none of the 12 randomly selected bolts is found to be defective, he concludes that the shipment is acceptable. If 10% of the bolts in the population are defective, what is the probability that none of the selected bolts are defective? [5]

\[ N = 500, \ M = 50, \ m = 12, \ x = 0 \]

\[ P(x = 0) = \frac{(500)(450)}{(12)} = \frac{(450)}{(12)} = 0.278 \]

10. The number of buses that arrive at a bus stop in T minutes is a Poisson random variable Y with expected value \( \lambda T \).

a. What is the PMF of Y, the number of buses that arrive in T minutes? [5]

\[ P(y = y) = \frac{e^{-\lambda T} (\lambda T)^y}{y!}, \ y = 0, 1, 2, \ldots \]

b. What is the probability that in a two-minute interval, three buses will arrive? [5]

\[ P(y = 3) = \frac{(2)^3 e^{-2}}{3!} = 0.0072 \]

c. What is the probability of no buses arriving in a 10-minute interval? [5]

\[ P(y = 0) = \frac{e^{-10/5} (10/5)^0}{0!} = e^{-2} = 0.135 \]

d. How much time should you allow so that with probability 0.99 at least one bus arrives? [10]

\[ P(y \geq 1) = 1 - P(y < 1) = 1 - P(y = 0) = 1 - e^{-T/5} \geq 0.99 \]

\[ \Rightarrow e^{-T/5} \leq 0.1 \]

\[ \Rightarrow -T/5 = \ln(0.1) \Rightarrow T = -5 \ln(0.1) = 23.02 \]